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Numerical and Asymptotic Aspects of Parabolic Cylinder Functions

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ABSTRACT

Several uniform asymptotics expansions of the Weber parabolic cylinder functions are considered, one group in terms of elementary functions, another group in terms of Airy functions. Starting point for the discussion are asymptotic expansions given earlier by F.W.J. Olver. Some of his results are modified to improve the asymptotic properties and to enlarge the intervals for using the expansions in numerical algorithms. Olver's results are obtained from the differential equation of the parabolic cylinder functions; we mention how modified expansions can be obtained from integral representations. Numerical tests are given for three expansions in terms of elementary functions. In this paper only real values of the parameters will be considered.

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Keywords and Phrases: parabolic cylinder functions, uniform asymptotic expansion, Airy-type expansions, numerical evaluation of special functions.

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1. Introduction

The solutions of the differential equation

$$\frac{d^2y}{dz^2} - \left(\tfrac{1}{4}z^2 + a\right) y = 0 \tag{1.1}$$

are associated with the parabolic cylinder in harmonic analysis; see [20]. The solutions are called parabolic cylinder functions and are entire functions of z. Many properties in connection with physical applications are given in [4].

As in [1], Chapter 19, we denote two standard solutions of (1.1) by $U(a,z), V(a,z)$. These solutions are given by the representations

$$\begin{aligned}
U(a,z) &= \sqrt{\pi}2^{-\frac{1}{2}a}\left[\frac{2^{-\frac{1}{4}}\,y_1(a,z)}{\Gamma(\frac{3}{4}+\frac{1}{2}a)} - \frac{2^{\frac{1}{4}}\,y_2(a,z)}{\Gamma(\frac{1}{4}+\frac{1}{2}a)}\right],\\
V(a,z) &= \frac{\sqrt{\pi}2^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2}-a)}\left[\tan\pi\left(\tfrac{1}{2}a+\tfrac{1}{4}\right)\frac{2^{-\frac{1}{4}}\,y_1(a,z)}{\Gamma(\frac{3}{4}+\frac{1}{2}a)} + \cot\pi\left(\tfrac{1}{2}a+\tfrac{1}{4}\right)\frac{2^{\frac{1}{4}}\,y_2(a,z)}{\Gamma(\frac{1}{4}+\frac{1}{2}a)}\right],
\end{aligned} \tag{1.2}$$

where

$$
\begin{aligned}
y_1(a,z) &= e^{-\frac14 z^2}\,{}_1F_1\left(-\tfrac12 a+\tfrac14,\tfrac12;-\tfrac12 z^2\right) = e^{\frac14 z^2}\,{}_1F_1\left(\tfrac12 a+\tfrac14,\tfrac12;\tfrac12 z^2\right),\\
y_2(a,z) &= z e^{-\frac14 z^2}\,{}_1F_1\left(\tfrac12 a+\tfrac34,\tfrac32;\tfrac12 z^2\right) = z e^{\frac14 z^2}\,{}_1F_1\left(-\tfrac12 a+\tfrac34,\tfrac32;-\tfrac12 z^2\right)
\end{aligned}
\tag{1.3}
$$

and the confluent hypergeometric function is defined by

$$
{}_1F_1(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!},
\tag{1.4}
$$

with $(a)_n = \Gamma(a+n)/\Gamma(a), n = 0,1,2,\ldots$.

Another notation found in the literature is

$$
D_\nu(z) = U(-\nu-\tfrac12, z).
$$

There is a relation with the Hermite polynomials. We have

$$
\begin{aligned}
U\left(-n-\tfrac12, z\right) &= 2^{-n/2} e^{-\frac14 z^2} H_n(z/\sqrt{2}),\\
V\left(n+\tfrac12, z\right) &= 2^{-n/2} e^{\frac14 z^2} (-i)^n H_n(iz/\sqrt{2}).
\end{aligned}
\tag{1.5}
$$

Other special cases are error functions and Fresnel integrals.

The Wronskian relation between $U(a,z)$ and $V(a,z)$ reads:

$$
U(a,z)V'(a,z) - U'(a,z)V(a,z) = \sqrt{2/\pi},
\tag{1.6}
$$

which shows that $U(a,z)$ and $V(a,z)$ are independent solutions of (1.1) for all values of a. Other relations are

$$
\begin{aligned}
U(a,z) &= \frac{\pi}{\cos^2\pi a\,\Gamma(\frac12+a)}\left[V(a,-z) - \sin\pi a\, V(a,z)\right],\\
V(a,z) &= \frac{\Gamma(\frac12+a)}{\pi}\left[\sin\pi a\, U(a,z) + U(a,-z)\right].
\end{aligned}
\tag{1.7}
$$

The functions $y_1(a,z)$ and $y_2(a,z)$ are the simplest even and odd solutions of (1.1) and the Wronskian of this pair equals 1. From a numerical point of view, the pair $\{y_1, y_2\}$ is not a satisfactory pair [8], because they have almost the same asymptotic behavior at infinity. The behavior of $U(a,z)$ and $V(a,z)$ is, for large positive z and $z \gg |a|$:

$$
\begin{aligned}
U(a,z) &= e^{-\frac14 z^2} z^{-a-\frac12}\left[1+\mathcal{O}\left(z^{-2}\right)\right],\\
V(a,z) &= \sqrt{2/\pi}\, e^{\frac14 z^2} z^{a-\frac12}\left[1+\mathcal{O}\left(z^{-2}\right)\right].
\end{aligned}
\tag{1.8}
$$

Clearly, numerical computations of $U(a,z)$ that are based on the representations in (1.2) should be done with great care, because of the loss of accuracy if z becomes large.

Equation (1.1) has two turning points at $\pm 2\sqrt{-a}$. For real parameters they become important if a is negative, and the asymptotic behavior of the solutions of (1.1) as $a \to -\infty$ changes significantly if z crosses the turning points. At these points Airy functions are needed. By changing the parameters it is not difficult to verify that $U(-\frac{1}{2}\mu^2, \mu t\sqrt{2})$ and $V(-\frac{1}{2}\mu^2, \mu t\sqrt{2})$ satisfy the simple equation

$$\frac{d^2y}{dt^2} - \mu^4\left(t^2-1\right)y = 0, \tag{1.9}$$

with turning points at $t = \pm 1$. For physical applications, negative $a-$values are most important (with special case the real Hermite polynomials, see (1.5)). For positive a we can use the notation $U(\frac{1}{2}\mu^2, \mu t\sqrt{2})$ and $V(\frac{1}{2}\mu^2, \mu t\sqrt{2})$, which satisfy the equation

$$\frac{d^2y}{dt^2} - \mu^4\left(t^2+1\right)y = 0. \tag{1.10}$$

The purpose of this paper is to give several asymptotic expansions of $U(a,z)$ and $V(a,z)$ that can be used for computing these functions for the case that at least one of the real parameters is large. In [10] an extensive collection of asymptotic expansions for the parabolic cylinder functions as $|a| \to \infty$ has been derived from the differential equation (1.1). The expansions are valid for complex values of the parameters and are given in terms of elementary functions and Airy functions. In Section 2 we mention several expansions in terms of elementary functions derived by Olver and modify some his results in order to improve the asymptotic properties of the expansions, to enlarge the intervals for using the expansions in numerical algorithms, and to get new recursion relations for the coefficients of the expansions. In Section 3 we give similar results for expansions in terms of Airy functions. In Section 4 we give information on how to obtain the modified results by using integral representations of the parabolic cylinder functions. Finally we give numerical tests for three expansions in terms of elementary functions, with a few number of terms in the expansions. Only real parameters are considered in this paper.

1.1 Recent literature on numerical algorithms

Recent papers on numerical algorithms for the parabolic cylinder functions are given in [14] (Fortran; $U(n,x)$ for natural n and positive x) and [13] (Fortran; $U(a,x), V(a,x)$, a integer and half-integer and $x \geq 0$). The methods are based on backward and forward recursion.

[2] gives programs in $\mathcal{C}$ for $U(a,x), V(a,x)$, and uses representations in terms of the confluent hypergeometric functions and asymptotic expressions, including those involving Airy functions. [23] gives Fortran programs for computing $U(a,z), V(a,z)$ with real orders and real argument, and for half-integer order and complex argument, The methods are based on recursions, Maclaurin series and asymptotic expansions. They refer also to [3] for the evaluation of $U(-ia, ze^{\frac{1}{4}\pi i})$ for real a and z (this function is a solution of the differential equation $y'' + (\frac{1}{4}z^2 - a)y = 0$). [19] uses series expansions and numerical quadrature; Fortran and $\mathcal{C}$ programs are given, and *Mathematica* cells to make graphical and numerical objects.

Maple [5] has algorithms for hypergeometric functions, which can be used in (1.2) and (1.3). *Mathematica* [21] refers for the parabolic cylinder functions to their programs for the hypergeometric functions and the same advice is given in [12].

2. Expansions in terms of elementary functions

2.1 The case $a \le 0, \quad z > 2\sqrt{-a}, \quad -a + z \gg 0.$

Olver's expansions in terms of elementary functions are all based on the expansion O-(4.3) [1]

$$U\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) \sim \frac{g(\mu)\, e^{-\mu^2\xi}}{(t^2-1)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{\mathcal{A}_s(t)}{\mu^{2s}}, \tag{2.1}$$

as $\mu \to \infty$, uniformly with respect to $t \in [1+\varepsilon, \infty)$; ε is a small positive number and ξ is given by

$$\xi = \tfrac{1}{2} t\sqrt{t^2-1} - \tfrac{1}{2} \ln\left[t + \sqrt{t^2-1}\right]. \tag{2.2}$$

The expansion is valid for complex parameters in large domains of the $\mu-$ and $t-$planes; details on these domains are not given here.

The coefficients $\mathcal{A}_s(t)$ are given by the recursion relation

$$\mathcal{A}_{s+1}(t) = \tfrac{1}{2} \frac{1}{\sqrt{t^2-1}} \frac{d\mathcal{A}_s(t)}{dt} + \tfrac{1}{8} \int_{c_{s+1}}^{t} \frac{3u^2+2}{(u^2-1)^{\frac{5}{2}}} \mathcal{A}_s(u)\, du, \quad \mathcal{A}_0(t) = 1, \tag{2.3}$$

where the constants c_s can be chosen in combination with the choice of $g(\mu)$. Olver chose the constants such that

$$\mathcal{A}_s(t) = \frac{u_s(t)}{(t^2-1)^{3s/2}}, \tag{2.4}$$

where the $u_s(t)$ are polynomials in t of degree $3s$, (s odd), $3s-2$ (s even, $s \ge 2$). The first few are

$$u_0(t) = 1, \quad u_1(t) = \frac{t(t^2-6)}{24}, \quad u_2(t) = \frac{-9t^4 + 249t^2 + 145}{1152},$$

and they satisfy the recurrence relation

$$(t^2-1)u_s'(t) - 3stu_s(t) = r_{s-1}(t), \tag{2.5}$$

where

$$8r_s(t) = (3t^2+2)u_s(t) - 12(s+1)tr_{s-1}(t) + 4(t^2-1)r_{s-1}'(t).$$

The quantity $g(\mu)$ in (2.1) is only available in the form of an asymptotic expansion:

$$g(\mu) \sim h(\mu) \left(\sum_{s=0}^{\infty} \frac{g_s}{\mu^{2s}}\right)^{-1}, \tag{2.6}$$

where

$$h(\mu) = 2^{-\frac{1}{4}\mu^2 - \frac{1}{4}} e^{-\frac{1}{4}\mu^2} \mu^{\frac{1}{2}\mu^2 - \frac{1}{2}}, \tag{2.7}$$

$$g_0 = 1, \quad g_1 = \tfrac{1}{24}, \quad g_3 = -\tfrac{2021}{207360}, \quad g_{2s} = 0 \ (s = 1, 2, \dots),$$

and in general

$$g_s = \lim_{t\to\infty} \mathcal{A}_s(t). \tag{2.8}$$

[1] We refer to Olver's equations by writing O-(4.3), and so on.

2.1.1 Modified expansions We modify the expansion in (2.1) by writing

$$U\left(-\tfrac{1}{2}\mu^2,\mu t\sqrt{2}\right)=\frac{h(\mu)\,e^{-\mu^2\xi}}{(t^2-1)^{\frac{1}{4}}}F_\mu(t),\quad F_\mu(t)\sim\sum_{s=0}^{\infty}\frac{\phi_s(\tau)}{\mu^{2s}},\tag{2.9}$$

where $h(\mu)$ and ξ are as before, and

$$\tau=\tfrac{1}{2}\left[\frac{t}{\sqrt{t^2-1}}-1\right].\tag{2.10}$$

The coefficients $\phi_s(\tau)$ are polynomials in τ, $\phi_0(\tau)=1$, and are given by the recursion

$$\phi_{s+1}(\tau)=-4\tau^2(\tau+1)^2\frac{d}{d\tau}\phi_s(\tau)-\tfrac{1}{4}\int_0^\tau\left(20u^2+20u+3\right)\phi_s(u)\,du.\tag{2.11}$$

This recursion follows from (2.3) by substituting $t=(\tau+\frac{1}{2})/\sqrt{\tau(\tau+1)}$, which is the inverse of the relation in (2.10). Explicitly,

$$\begin{aligned}
\phi_0(\tau)&=1,\\
\phi_1(\tau)&=-\frac{\tau}{12}(20\tau^2+30\tau+9),\\
\phi_2(\tau)&=\frac{\tau^2}{288}(6160\tau^4+18480\tau^3+19404\tau^2+8028\tau+945),\\
\phi_3(\tau)&=-\frac{\tau^3}{51840}(27227200\tau^6+122522400\tau^5+220540320\tau^4+\\
&\qquad 200166120\tau^3+94064328\tau^2+20545650\tau+1403325),
\end{aligned}\tag{2.12}$$

where τ is given in (2.10). Observe that $\lim_{t\to\infty}\tau(t)=0$ and that all shown coefficients $\phi_s(\tau)$ vanish at infinity for $s>0$. These properties of $\phi_s(\tau)$ follow by taking different constants c_s than Olver did in (2.3). In fact we have the relation

$$\sum_{s=0}^{\infty}\frac{g_s}{\mu^{2s}}\sum_{s=0}^{\infty}\frac{\phi_s(\tau)}{\mu^{2s}}\sim\sum_{s=0}^{\infty}\frac{u_s(t)}{(t^2-1)^{\frac{3}{2}s}\mu^{2s}},$$

where the first series appears in (2.6). Explicitly,

$$u_s(t)=\left(t^2-1\right)^{\frac{3}{2}s}\sum_{j=0}^{s}g_{s-j}\phi_j(\tau).\tag{2.13}$$

The relation (2.13) can easily be verified for the early coefficients, but it holds because of the unicity of Poincaré-type asymptotic expansions.

The expansion in (2.9) has several advantages compared with (2.1).

- In the recursion relation (2.5), both u_s and u_s' occur in the left-hand side. By using computer algebra it is not difficult to compute any number of coefficients u_s, but the relation for the polynomials $\phi_s(\tau)$ is simpler than (2.5), with this respect.
- The quantity $h(\mu)$ in (2.9) is defined as an exact relation, and not, as $g(\mu)$ in (2.1), by an asymptotic expansion (cf. (2.6)).

- Most important, the expansion in (2.9) has a double asymptotic property: it holds if one or both parameters t and μ are large, and not only if μ is large.

For the function $V(a,z)$ we have

$$V\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{e^{\mu^2\xi}}{\mu\sqrt{\pi}\,h(\mu)\,(t^2-1)^{\frac{1}{4}}}\,P_\mu(t), \quad P_\mu(t) \sim \sum_{s=0}^{\infty}(-1)^s\frac{\phi_s(\tau)}{\mu^{2s}}, \tag{2.14}$$

where the $\phi_s(\tau)$ are the same as in (2.9). This expansion is a modification of O-(11.19) (see also O-(2.12)).

For the derivatives we can use the identities

$$\begin{aligned} \frac{d}{dt}\frac{e^{-\mu^2\xi}}{(t^2-1)^{\frac{1}{4}}}F_\mu(t) &= -\mu^2(t^2-1)^{\frac{1}{4}}e^{-\mu^2\xi}\,G_\mu(t), \quad G_\mu(t) \sim \sum_{s=0}^{\infty}\frac{\psi_s(\tau)}{\mu^{2s}}, \\ \frac{d}{dt}\frac{e^{+\mu^2\xi}}{(t^2-1)^{\frac{1}{4}}}P_\mu(t) &= +\mu^2(t^2-1)^{\frac{1}{4}}e^{+\mu^2\xi}\,Q_\mu(t), \quad Q_\mu(t) \sim \sum_{s=0}^{\infty}(-1)^s\frac{\psi_s(\tau)}{\mu^{2s}}. \end{aligned} \tag{2.15}$$

The coefficients ψ_s can be obtained from the relation

$$\psi_s(t) = \phi_s(\tau) + 2\tau(\tau+1)(2\tau+1)\phi_{s-1}(\tau) + 8\tau^2(\tau+1)^2\frac{d\phi_{s-1}(\tau)}{d\tau}, \tag{2.16}$$

$s = 0, 1, 2, \ldots$. The first few are

$$\begin{aligned} \psi_0(t) &= 1, \\ \psi_1(t) &= \frac{\tau}{12}(28\tau^2+42\tau+15), v \\ \psi_2(t) &= -\frac{\tau^2}{288}(7280\tau^4+21840\tau^3+23028\tau^2+9684\tau+1215), \\ \psi_3(t) &= \frac{\tau^3}{51840}(30430400\tau^6+136936800\tau^5+246708000\tau^4+ \\ &\qquad 224494200\tau^3+106122312\tau^2+23489190\tau+1658475). \end{aligned} \tag{2.17}$$

This gives the modifications (see O-(4.13))

$$U'\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = -\frac{\mu}{\sqrt{2}}h(\mu)(t^2-1)^{\frac{1}{4}}e^{-\mu^2\xi}\,G_\mu(t), \quad G_\mu(t) \sim \sum_{s=0}^{\infty}\frac{\psi_s(\tau)}{\mu^{2s}} \tag{2.18}$$

and

$$V'\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{(t^2-1)^{\frac{1}{4}}e^{\mu^2\xi}}{\sqrt{2\pi}h(\mu)}\,Q_\mu(t), \quad Q_\mu(t) \sim \sum_{s=0}^{\infty}(-1)^s\frac{\psi_s(\tau)}{\mu^{2s}}. \tag{2.19}$$

Remark 2.1. The functions $F_\mu(t), G_\mu(t), P_\mu(t)$ and $Q_\mu(t)$ introduced in the asymptotic representations satisfy the following exact relation:

$$F_\mu(t)\,Q_\mu(t) + G_\mu(t)\,P_\mu(t) = 2. \tag{2.20}$$

This follows from the Wronskian relation (1.6). The relation in (2.20) provides a convenient possibility for checking the accuracy in numerical algorithms that use the asymptotic expansions of $F_\mu(t), G_\mu(t), P_\mu(t)$ and $Q_\mu(t)$.

2.2 The case $a \le 0, \quad z < -2\sqrt{-a}, \quad -a - z \gg 0.$
For this case we mention the modification of O-(11.16). That is, for $t \ge 1 + \varepsilon$ we have the representations

$$\begin{aligned} U\left(-\tfrac{1}{2}\mu^2, -\mu t\sqrt{2}\right) &= \frac{h(\mu)}{(t^2-1)^{\frac14}} \quad \times \\ &\left[\sin\left(\tfrac{1}{2}\pi\mu^2\right) e^{-\mu^2\xi}\, F_\mu(t) + \frac{\Gamma(\frac12+\frac12\mu^2)\cos\left(\frac12\pi\mu^2\right)}{\mu\,\sqrt{\pi}\,h^2(\mu)} e^{\mu^2\xi}\, P_\mu(t)\right], \end{aligned} \tag{2.21}$$

where $F_\mu(t)$ and $P_\mu(t)$ have the expansions given in (2.9) and (2.14), respectively. An expansion for $V(-\frac12\mu^2, -\mu t\sqrt{2})$ follows from the second line in (1.7), (2.9) and (2.21). A few manipulations give

$$\begin{aligned} V\left(-\tfrac{1}{2}\mu^2, -\mu t\sqrt{2}\right) &= \frac{h(\mu)}{(t^2-1)^{\frac14}\Gamma(\frac12+\frac12\mu^2)} \quad \times \\ &\left[\cos\left(\tfrac{1}{2}\pi\mu^2\right) e^{-\mu^2\xi}\, F_\mu(t) - \frac{\Gamma(\frac12+\frac12\mu^2)\sin\left(\frac12\pi\mu^2\right)}{\mu\,\sqrt{\pi}\,h^2(\mu)} e^{\mu^2\xi}\, P_\mu(t)\right]. \end{aligned} \tag{2.22}$$

Expansions for the derivatives follow from the identities in (2.15). If $a = -\frac12\mu^2 = -n - \frac12, n = 0, 1, 2, \ldots,$ the cosine in (2.21) vanishes, and, hence, the dominant part vanishes. This is the Hermite case, cf. (1.5).

2.3 The case $a \ll 0, \quad -2\sqrt{-a} < z < 2\sqrt{-a}.$
For negative a and $-1 < t < 1$ the expansions are essentially different, because now oscillations with respect to t occur. We have (O-(5.11) and O-(5.23))

$$\begin{aligned} U\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) &\sim \frac{2g(\mu)}{(1-t^2)^{\frac14}} \quad \times \\ &\left[\cos\left(\mu^2\eta - \tfrac14\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s u_{2s}(t)}{(1-t^2)^{3s}\mu^{4s}} - \sin\left(\mu^2\eta - \tfrac14\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s u_{2s+1}(t)}{(1-t^2)^{3s+\frac32}\mu^{4s+2}}\right], \end{aligned} \tag{2.23}$$

with $u_s(t)$ defined in (2.5) and $g(\mu)$ in (2.6), and

$$\begin{aligned} U'\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) &\sim \mu\sqrt{2}g(\mu)(1-t^2)^{\frac14} \quad \times \\ &\left[\sin\left(\mu^2\eta - \tfrac14\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s v_{2s}(t)}{(1-t^2)^{3s}\mu^{4s}} + \cos\left(\mu^2\eta - \tfrac14\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s v_{2s+1}(t)}{(1-t^2)^{3s+\frac32}\mu^{4s+2}}\right], \end{aligned} \tag{2.24}$$

as $\mu \to \infty$, uniformly with respect to $|t| \le 1 - \varepsilon$, where the coefficients v_s are given by (see O-(4.15))

$$v_s(t) = u_s(t) + \tfrac12 t u_{s-1}(t) - r_{s-2}(t), \tag{2.25}$$

and

$$\eta = \tfrac{1}{2}\arccos t - \tfrac{1}{2}t\sqrt{1-t^2}. \tag{2.26}$$

For the function $V(a,z)$ we have (O-(11.20) and O(2.12))

$$\begin{aligned}
V\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) &\sim \frac{2g(\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)(1-t^2)^{\frac{1}{4}}} \quad \times \\
&\left[\cos\left(\mu^2\eta + \tfrac{1}{4}\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s u_{2s}(t)}{(1-t^2)^{3s}\mu^{4s}} - \sin\left(\mu^2\eta+\tfrac{1}{4}\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s u_{2s+1}(t)}{(1-t^2)^{3s+\frac{3}{2}}\mu^{4s+2}}\right], \\
V'\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) &\sim \frac{\mu\sqrt{2}g(\mu)(1-t^2)^{\frac{1}{4}}}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)} \quad \times \\
&\left[\sin\left(\mu^2\eta + \tfrac{1}{4}\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s v_{2s}(t)}{(1-t^2)^{3s}\mu^{4s}} + \cos\left(\mu^2\eta+\tfrac{1}{4}\pi\right)\sum_{s=0}^{\infty}\frac{(-1)^s v_{2s+1}(t)}{(1-t^2)^{3s+\frac{3}{2}}\mu^{4s+2}}\right],
\end{aligned} \tag{2.27}$$

By using the Wronskian relation (1.6) it follows that we have the following asymptotic identity

$$\begin{aligned}
&\sum_{s=0}^{\infty}\frac{(-1)^s u_{2s}(t)}{(1-t^2)^{3s}\mu^{4s}}\sum_{s=0}^{\infty}\frac{(-1)^s v_{2s}(t)}{(1-t^2)^{3s}\mu^{4s}} + \\
&\sum_{s=0}^{\infty}\frac{(-1)^s u_{2s+1}(t)}{(1-t^2)^{3s+\frac{3}{2}}\mu^{4s+2}}\sum_{s=0}^{\infty}\frac{(-1)^s v_{2s+1}(t)}{(1-t^2)^{3s+\frac{3}{2}}\mu^{4s+2}} \\
&\sim \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)}{2\mu\sqrt{\pi}g^2(\mu)} \sim 1 - \frac{1}{576\mu^4} + \frac{2021}{2488320\mu^8} + \ldots\,.
\end{aligned} \tag{2.28}$$

2.3.1 Modified expansions We can give modified versions based on our earlier modifications, with $g(\mu)$ replaced with $h(\mu)$, and so on. Because in the present case t belongs to a finite domain, the modified expansions don't have the double asymptotic property. We prefer Olver's versions for this case.

This completes the description of $U(a,z), U'(a,z), V(a,z), V'(a,z)$ in terms of elementary functions for negative values of a.

2.4 The case $a \ge 0, \quad z \ge 0, \quad a+z \gg 0.$

For positive values of a the asymptotic behavior is rather simple because no oscillations occur now. The results follow from Olver's expansions O-(11.10) and O-(11.12). The modified forms are

$$U\left(\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{\widetilde{h}(\mu)\, e^{-\mu^2\widetilde{\xi}}}{(t^2+1)^{\frac{1}{4}}}\widetilde{F}_\mu(t), \quad \widetilde{F}_\mu(t) \sim \sum_{s=0}^{\infty}(-1)^s\,\frac{\phi_s(\widetilde{\tau})}{\mu^{2s}}, \tag{2.29}$$

where

$$\widetilde{\xi} = \tfrac{1}{2}\left[t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\right], \tag{2.30}$$

$$\widetilde{h}(\mu) = e^{\frac{1}{4}\mu^2} \mu^{-\frac{1}{2}\mu^2-\frac{1}{2}} 2^{\frac{1}{4}\mu^2-\frac{1}{4}}. \tag{2.31}$$

The coefficients ϕ_s in (2.29) are the same as in (2.9), with τ replaced by

$$\widetilde{\tau} = \tfrac{1}{2}\left[\frac{t}{\sqrt{1+t^2}} - 1\right]. \tag{2.32}$$

For the derivative we have

$$U'\left(\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = -\frac{1}{\sqrt{2}}\mu\widetilde{h}(\mu)(1+t^2)^{\frac{1}{4}}\, e^{-\mu^2\widetilde{\xi}}\widetilde{G}_a(z), \quad \widetilde{G}_a(z) \sim \sum_{s=0}^{\infty}(-1)^s\,\frac{\psi_s(\widetilde{\tau})}{\mu^{2s}}, \tag{2.33}$$

where $\psi_s(\widetilde{\tau})$ is given in (2.16), with $\widetilde{\tau}$ defined in (2.32).

2.5 The case $\quad a \ge 0, \quad z \le 0, \quad a - z \gg 0.$
Olver's expansion O-(11.10) and O(11.12) cover both cases $z \ge 0$ and $z \le 0$. We have the modified expansions

$$\begin{aligned} U\left(\tfrac{1}{2}\mu^2, -\mu t\sqrt{2}\right) &= \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)}\frac{h(\mu)e^{\mu^2\widetilde{\xi}}}{(1+t^2)^{\frac{1}{4}}}\,\widetilde{P}_\mu(t), \\ U'\left(\tfrac{1}{2}\mu^2, -\mu t\sqrt{2}\right) &= -\frac{\mu}{\sqrt{2}}\frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)}h(\mu)e^{\mu^2\widetilde{\xi}}(1+t^2)^{\frac{1}{4}}\,\widetilde{Q}_\mu(t), \end{aligned} \tag{2.34}$$

where

$$\widetilde{P}_\mu(t) \sim \sum_{s=0}^{\infty}\frac{\phi_s(\widetilde{\tau})}{\mu^{2s}}, \quad \widetilde{Q}_\mu(t) \sim \sum_{s=0}^{\infty}\frac{\psi_s(\widetilde{\tau})}{\mu^{2s}}.$$

In Section 4.1.2 we give details on the derivation of these expansions.
Remark 2.2. By using the second relation in (1.7), the representations for $V(a,z)$ and $V'(a,z)$ for positive a can be obtained from the results for $U(a,z)$ and $U'(a,z)$ in (2.29), (2.33) and (2.34).
Remark 2.3. The functions $\widetilde{F}_\mu(t), \widetilde{G}_\mu(t), \widetilde{P}_\mu(t)$ and $\widetilde{Q}_\mu(t)$ introduced in (2.29), (2.32) and (2.34) satisfy the following exact relation:

$$\widetilde{F}_\mu(t)\widetilde{Q}_\mu(t) + \widetilde{G}_\mu(t)\widetilde{P}_\mu(t) = 2. \tag{2.35}$$

This follows from the Wronskian relation

$$U(a,z)U'(a,-z) + U'(a,z)U(a,-z) = -\frac{\sqrt{2\pi}}{\Gamma(a+\frac{1}{2})}.$$

See also Remark 2.1.
Remark 2.4. The expansions of Sections 2.4 and 2.5 have the double asymptotic property: they are valid if the $a + |z| \to \infty$. In Sections 2.4 and 2.5 we consider the cases $z \ge 0$ and $z \le 0$, respectively, as two separate cases. Olver's corresponding expansions O-(11.10) and O-(11.12) cover both cases and are valid for $-\infty < t < \infty$. As always, in Olver's expansions large values of μ are needed, whatever the size of t.

In Figure 1 we show the domains in the $t, a-$plane where the various expansions of $U(a,z)$ of this section are valid.

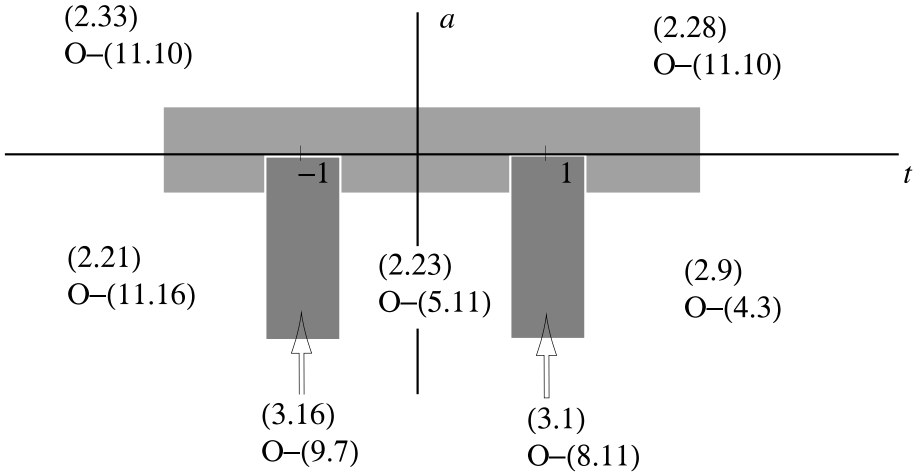

Figure 1. Regions for the modified asymptotic expansions of $U(a,z)$ given in Section 2 and the Airy-type expansions of Section 3 (which are valid in much larger domains than those indicated by the arrows).

3. Expansions in terms of Airy functions

The Airy-type expansions are needed if z runs through an interval containing one of the turning points $\pm 2\sqrt{-a}$, that is, $t = \pm 1$.

3.1 The case $\quad a \ll 0, \quad z \ge 0.$

We summarize the basic results O-(8.11), O-(8.15) and O-(11.22) (see also O-(2.12)):

$$U\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = 2\pi^{\frac{1}{2}}\mu^{\frac{1}{3}} g(\mu)\phi(\zeta)\left[\mathrm{Ai}(\mu^{\frac{4}{3}}\zeta)\, A_\mu(\zeta) + \frac{\mathrm{Ai}'(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{8}{3}}}\, B_\mu(\zeta)\right], \tag{3.1}$$

$$U'\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{(2\pi)^{\frac{1}{2}}\mu^{\frac{2}{3}} g(\mu)}{\phi(\zeta)}\left[\frac{\mathrm{Ai}(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{4}{3}}}\, C_\mu(\zeta) + \mathrm{Ai}'(\mu^{\frac{4}{3}}\zeta)\, D_\mu(\zeta)\right], \tag{3.2}$$

$$V\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{2\pi^{\frac{1}{2}}\mu^{\frac{1}{3}} g(\mu)\phi(\zeta)}{\Gamma(\frac{1}{2} + \frac{1}{2}\mu^2)}\left[\mathrm{Bi}(\mu^{\frac{4}{3}}\zeta)\, A_\mu(\zeta) + \frac{\mathrm{Bi}'(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{8}{3}}}\, B_\mu(\zeta)\right], \tag{3.3}$$

$$V'\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{(2\pi)^{\frac{1}{2}}\mu^{\frac{2}{3}} g(\mu)}{\phi(\zeta)\Gamma(\frac{1}{2} + \frac{1}{2}\mu^2)}\left[\frac{\mathrm{Bi}(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{4}{3}}}\, C_\mu(\zeta) + \mathrm{Bi}'(\mu^{\frac{4}{3}}\zeta)\, D_\mu(\zeta)\right]. \tag{3.4}$$

The coefficient functions $A_\mu(\zeta), B_\mu(\zeta), C_\mu(\zeta)$ and $D_\mu(\zeta)$ have the following asymptotic expansions

$$A_\mu(\zeta) \sim \sum_{s=0}^{\infty} \frac{a_s(\zeta)}{\mu^{4s}}, \quad B_\mu(\zeta) \sim \sum_{s=0}^{\infty} \frac{b_s(\zeta)}{\mu^{4s}}, \tag{3.5}$$

$$C_\mu(\zeta) \sim \sum_{s=0}^{\infty} \frac{c_s(\zeta)}{\mu^{4s}}, \quad D_\mu(\zeta) \sim \sum_{s=0}^{\infty} \frac{d_s(\zeta)}{\mu^{4s}}, \tag{3.6}$$

as $\mu \to \infty$, uniformly with respect to $t \geq -1+\delta$, where δ is a small fixed positive number. The quantity ζ is defined by

$$\begin{array}{lll} \frac{2}{3}(-\zeta)^{\frac{3}{2}} & = \eta(t), & -1 < t \leq 1, \quad (\zeta \leq 0) \\ \frac{2}{3}\zeta^{\frac{3}{2}} & = \xi(t), & 1 \leq t, \quad (\zeta \geq 0) \end{array} \tag{3.7}$$

where η, ξ follow from (2.26), (2.2), respectively, and

$$\phi(\zeta) = \left(\frac{\zeta}{t^2-1}\right)^{\frac{1}{4}}. \tag{3.8}$$

The function $\zeta(t)$ is real for $t > -1$ and analytic at $t = 1$. We can invert $\zeta(t)$ into $t(\zeta)$, and obtain

$$t = 1 + 2^{-1/3}\zeta - \tfrac{1}{10}2^{-2/3}\zeta^2 + \tfrac{11}{700}\zeta^3 + \dots \ .$$

The function $g(\mu)$ has the expansion given in (2.6) and the coefficients $a_s(\zeta), b_s(\zeta)$ are given by

$$a_s(\zeta) = \sum_{m=0}^{2s} \beta_m \zeta^{-\frac{3}{2}m} \mathcal{A}_{2s-m}(t) \quad \sqrt{\zeta} b_s(\zeta) = -\sum_{m=0}^{2s+1} \alpha_m \zeta^{-\frac{3}{2}m} \mathcal{A}_{2s-m+1}(t), \tag{3.9}$$

where $\mathcal{A}_s(t)$ are used in (2.1), $\alpha_0 = 1$ and

$$\alpha_m = \frac{(2m+1)\,(2m+3)\cdots(6m-1)}{m!\,(144)^m}, \quad \beta_m = -\frac{6m+1}{6m-1}\,a_m. \tag{3.10}$$

A recursion for α_m reads:

$$\alpha_{m+1} = \alpha_m \frac{(6m+5)(6m+3)(6m+1)}{144\,(m+1)(2m+1)}, \quad m = 0, 1, 2, \dots \ .$$

The numbers α_m, β_m occur in the asymptotic expansions of the Airy functions, and the relations in (3.9) follow from solving (3.1) and (3.3) for $A_\mu(\zeta)$ and $B_\mu(\zeta)$, expanding the Airy functions (assuming that ζ is bounded away from 0) and by using (2.1) and a similar result for $V(a, z)$ (O-(11.16) and O-(2.12)).

For negative values of ζ (i.e., $-1 < t < 1$) we can use (O-(13.4))

$$\begin{array}{ll} a_s(\zeta) & = (-1)^s \sum_{m=0}^{2s} \beta_m (-\zeta)^{-\frac{3}{2}m} \widetilde{\mathcal{A}}_{2s-m}(t), \\ \sqrt{-\zeta} b_s(\zeta) & = (-1)^{s-1} \sum_{m=0}^{2s+1} \alpha_m (-\zeta)^{-\frac{3}{2}m} \widetilde{\mathcal{A}}_{2s-m+1}(t), \end{array} \tag{3.11}$$

where

$$\widetilde{\mathcal{A}}_s(t) = \frac{u_s(t)}{(1-t^2)^{\frac{3}{2}s}}.$$

The functions $C_\mu(\zeta)$ and $D_\mu(\zeta)$ of (3.2) and (3.4) are given by

$$C_\mu(\zeta) = \chi(\zeta)A_\mu(\zeta) + A'_\mu(\zeta) + \zeta B_\mu(\zeta), \quad D_\mu(\zeta) = A_\mu(\zeta) + \frac{1}{\mu^4}\left[\chi(\zeta)B_\mu(\zeta) + B'_\mu(\zeta)\right]. \tag{3.12}$$

The coefficients $c_s(\zeta)$ and $d_s(\zeta)$ in (3.6) are given by

$$c_s(\zeta) = \chi(\zeta)a_s(\zeta) + a'_s(\zeta) + \zeta b_s(\zeta), \quad d_s(\zeta) = a_s(\zeta) + \chi(\zeta)b_{s-1}(\zeta) + b'_{s-1}(\zeta), \tag{3.13}$$

where

$$\chi(\zeta) = \frac{\phi'(\zeta)}{\phi(\zeta)} = \frac{1-2t[\phi(\zeta)]^6}{4\zeta}, \tag{3.14}$$

with $\phi(\zeta)$ given in (3.8). Explicitly,

$$\frac{1}{\sqrt{\zeta}}c_s(\zeta) = -\sum_{m=0}^{2s+1}\beta_m\zeta^{-\frac{3}{2}m}\mathcal{B}_{2s-m+1}(\tau) \quad d_s(\zeta) = -\sum_{m=0}^{2s}\alpha_m\zeta^{-\frac{3}{2}m}\mathcal{B}_{2s-m}(\tau), \tag{3.15}$$

where $\mathcal{B}_s(\tau) = v_s(t)/(t^2-1)^{\frac{3}{2}s}$, with $v_s(t)$ defined in (2.25). Other versions of (3.15) are needed for negative values of ζ,i.e., if $-1 < t < 1$; see (3.11).

3.2 The case $a \ll 0, z \le 0$.
Near the other turning point $t = -1$ we can use the representations (O-(9.7))

$$\begin{aligned} U\left(-\tfrac{1}{2}\mu^2, -\mu t\sqrt{2}\right) = 2\pi^{\frac{1}{2}}\mu^{\frac{1}{3}}g(\mu)\phi(\zeta)\Bigg[\sin\left(\tfrac{1}{2}\pi\mu^2\right)\Bigg\{\mathrm{Ai}(\mu^{\frac{4}{3}}\zeta)A_\mu(\zeta) + \\ \frac{\mathrm{Ai}'(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{8}{3}}}B_\mu(\zeta)\Bigg\} + \cos\left(\tfrac{1}{2}\pi\mu^2\right)\Bigg\{\mathrm{Bi}(\mu^{\frac{4}{3}}\zeta)A_\mu(\zeta) + \frac{\mathrm{Bi}'(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{8}{3}}}B_\mu(\zeta)\Bigg\}\Bigg], \end{aligned} \tag{3.16}$$

as $\mu \to \infty$, uniformly with respect to $t \ge -1+\delta$, where δ is a small fixed positive number. Expansions for $V(a,z)$ follow from (3.1) and (3.16) and the second relation in (1.7). Results for the derivatives of $U(a,z)$ and $V(a,z)$ follow easily from the earlier results.

3.3 Modified forms of Olver's Airy-type expansions
Modified versions of the Airy-type expansions (3.1) – (3.4) can also be given. In the case of the expansions in terms of elementary functions our main motivation for introducing modified expansions was the double asymptotic property of these expansions. In the case of the Airy-type expansions the interesting domains for the parameter t, from a numerical point of view, are finite domains that contain the turning points ± 1. So, considering the expansions given so far, there is no need to have Airy-type expansions with the double asymptotic property; if μ remains finite and $|t| \gg 1$ we can use the expansions in terms of elementary functions. However, we have another interest in modified expansions in the case of Airy-type expansions. We explain this by first discussing a few properties of the coefficient functions $A_\mu(\zeta), B_\mu(\zeta), C_\mu(\zeta)$ and $D_\mu(\zeta)$.

By using the Wronskian relation (1.6) we can verify the relation

$$A_\mu(\zeta)D_\mu(\zeta) - \frac{1}{\mu^4}B_\mu(\zeta)C_\mu(\zeta) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)}{2\mu\sqrt{\pi}g^2(\mu)}, \tag{3.17}$$

where $g(\mu)$ is defined by means of an asymptotic expansion given in (2.6). By using the differential equation (O-(7.2))

$$\frac{d^2W}{d\zeta^2} = \left[\mu^4\zeta + \Psi(\zeta)\right] W, \tag{3.18}$$

where

$$\Psi(\zeta) = \frac{5}{16\zeta^2} - \frac{(3t^2+2)\zeta}{4(t^2-1)^3} = 2^{1/3}\left[-\tfrac{9}{280} + \tfrac{7}{150}2^{-1/3}\zeta - \tfrac{1359}{26950}2^{-2/3}\zeta^2 + \tfrac{196}{8125}\zeta^3 \ldots\right], \tag{3.19}$$

we can derive the following system of equations for the functions $A_\mu(\zeta), B_\mu(\zeta)$:

$$\begin{array}{ll} A'' + 2\zeta B' + B - \Psi(\zeta)A & = 0, \\ B'' + 2\mu^4 A' - \Psi(\zeta)B & = 0, \end{array} \tag{3.20}$$

where primes denote differentiation with respect to ζ. A Wronskian for this system follows by eliminating the terms with $\Psi(\zeta)$. This gives

$$2\mu^4 A'A + AB'' - A''B - 2\zeta B'B - B^2 = 0,$$

which can be integrated:

$$\mu^4 A_\mu^2(\zeta) + A_\mu(\zeta)B_\mu'(\zeta) - A_\mu'(\zeta)B_\mu(\zeta) - \zeta B_\mu^2(\zeta) = \mu^4 \frac{\Gamma(\frac12 + \frac12\mu^2)}{2\mu\sqrt{\pi}g^2(\mu)}, \tag{3.21}$$

where the quantity on the right-hand side follows from (3.17) and (3.12). It has the expansion

$$\mu^4\left[1 - \frac{1}{576\mu^4} + \frac{2021}{2488320\mu^8} + \ldots\right], \tag{3.22}$$

as follows from O-(2.22) and O-(5.21).

As mentioned before, the interesting domain of the Airy-type expansions given in this section is the domain that contains the turning point $t = 1$, or $\zeta = 0$. The representations of the coefficients of the expansions given in (3.9) cannot be used in numerical algorithms when $|\zeta|$ is small, unless we expand all relevant coefficients in powers of ζ. This is one way how to handle this problem numerically; see [18]. In that paper we have discussed another method that is based on a system like (3.20), with applications to Bessel functions. In that method the functions $A_\mu(\zeta)$ and $B_\mu(\zeta)$ are expanded in powers of ζ, for sufficiently small values of $|\zeta|$, say $|\zeta| \le 1$, and the Maclaurin coefficients are computed from (3.20) by recursion. A normalizing relation (the analogue of (3.21)) plays a crucial role in that algorithm. The method works quite well for relatively small values of a parameter (the order of the Bessel functions) that is the analogue of μ.

When we want to use this algorithm for the present case only large values of μ are allowed because the function $g(\mu)$ that is used in (3.1)–(3.4) and (3.21) is only defined for large values of μ. For this reason we give the modified versions of Olver's Airy-type expansions. The modified versions are more complicated than the Olver's expansions, because the analogues of the series in (3.5) and (3.6) are in powers of μ^{-2}, and not in powers of μ^{-4}. Hence, when we use these series for numerical computations we need more coefficients in the modified expansions, which is certainly not desirable from a numerical point of view, given the complexity of the coefficients in Airy-type expansions. However, in the algorithm based on Maclaurin expansions of the analogues of the coefficient functions $A_\mu(\zeta), B_\mu(\zeta), C_\mu(\zeta)$ and $D_\mu(\zeta)$ this point is of minor concern.

The modified expansions are the following:

$$U\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)\phi(\zeta)}{\mu^{\frac{2}{3}}h(\mu)}\left[\mathrm{Ai}(\mu^{\frac{4}{3}}\zeta)\,F_\mu(\zeta) + \frac{\mathrm{Ai}'(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{8}{3}}}\,G_\mu(\zeta)\right], \tag{3.23}$$

$$V\left(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{\phi(\zeta)}{\mu^{\frac{2}{3}}h(\mu)}\left[\mathrm{Bi}(\mu^{\frac{4}{3}}\zeta)\,F_\mu(\zeta) + \frac{\mathrm{Bi}'(\mu^{\frac{4}{3}}\zeta)}{\mu^{\frac{8}{3}}}\,G_\mu(\zeta)\right]. \tag{3.24}$$

The functions $F_\mu(\zeta)$ and $G_\mu(\zeta)$ have the following asymptotic expansions

$$F_\mu(\zeta) \sim \sum_{s=0}^{\infty}\frac{f_s(\zeta)}{\mu^{2s}}, \quad G_\mu(\zeta) \sim \sum_{s=0}^{\infty}\frac{g_s(\zeta)}{\mu^{2s}}. \tag{3.25}$$

The quantity ζ and the functions $\phi(\zeta)$ and $h(\mu)$ are as in Section 3.1. Comparing (3.23), (3.24) with (3.1), (3.3) we conclude that

$$F_\mu(\zeta) = H(\mu)A_\mu(\zeta), \quad G_\mu(\zeta) = H(\mu)B_\mu(\zeta), \quad H(\mu) = \frac{2\sqrt{\pi}\mu g(\mu)h(\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu^2)}. \tag{3.26}$$

The function $H(\mu)$ can be expanded (see O-(2.22), O-(2.27), O-(6.2) and (2.6))

$$H(\mu) \sim 1 + \tfrac{1}{2}\sum_{s=1}^{\infty}(-1)^s\frac{\gamma_s}{(\frac{1}{2}\mu^2)^s}, \tag{3.27}$$

where γ_s are the coefficients in the gamma function expansions

$$\Gamma(\tfrac{1}{2}+z) \sim \sqrt{2\pi}e^{-z}z^z\sum_{s=0}^{\infty}\frac{\gamma_s}{z^s}, \quad \frac{1}{\Gamma(\frac{1}{2}+z)} \sim \frac{e^z z^{-z}}{\sqrt{2\pi}}\sum_{s=0}^{\infty}(-1)^s\frac{\gamma_s}{z^s}. \tag{3.28}$$

The first few coefficients are

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{24}, \quad \gamma_2 = \frac{1}{1152}, \quad \gamma_3 = \frac{1003}{414720}.$$

The second expansion in (3.28) can be used in (3.26) to find relations between the coefficients $a_s(\zeta)$ and $b_s(\zeta)$ of (3.5) and of $f_s(\zeta)$ and $g_s(\zeta)$ of (3.25). That is,

$$f_0(\zeta) = 1, \quad f_1(\zeta) = \tfrac{1}{24}, \quad f_2(\zeta) = a_1(\zeta) + \tfrac{1}{576}, \quad f_3(\zeta) = \tfrac{1}{24}a_1(\zeta) - \tfrac{1003}{103680},$$

$$g_0(\zeta) = b_0(\zeta),\ g_1(\zeta) = \tfrac{1}{24}b_0(\zeta),\ g_2(\zeta) = b_1(\zeta) + \tfrac{1}{576}b_0(\zeta),\ g_3(\zeta) = \tfrac{1}{24}b_1(\zeta) - \tfrac{1003}{103680}b_0(\zeta).$$

The coefficients $f_s(\zeta), g_s(\zeta)$ can also be expressed in terms of the coefficients $\phi_s(\tau)$ that are introduced in (2.9) by deriving the analogues of (3.9).

The system of equations (3.20) remains the same:

$$
\begin{aligned}
F'' + 2\zeta G' + G - \Psi(\zeta)F &= 0,\\
G'' + 2\mu^4 F' - \Psi(\zeta)G &= 0,
\end{aligned}
\tag{3.29}
$$

and the Wronskian relation becomes

$$
\mu^4 F_\mu^2(\zeta) + F_\mu(\zeta)G'_\mu(\zeta) - F'_\mu(\zeta)G_\mu(\zeta) - \zeta G_\mu^2(\zeta) = \mu^4 \frac{2\sqrt{\pi}\mu h^2(\mu)}{\Gamma(\frac12+\frac12\mu^2)}. \tag{3.30}
$$

The right-hand side has the expansion (see (3.28) and (2.7)) $\mu^4 \sum_{s=0}^{\infty}(-1)^s\gamma_s/(\frac12\mu^2)^s$. Observe that ((3.30) is an exact relation, whereas (3.21) contains the function $g(\mu)$, of which only an asymptotic expansion is available.

3.4 Numerical aspects of the Airy-type expansions

In [18], (Section 4), we solved the system (3.29) (for the case of Bessel functions) by substituting Maclaurin series of $F(\zeta), G(\zeta)$ and $\Psi(\zeta)$. That is, we wrote

$$
F(\zeta) = \sum_{n=0}^{\infty} c_n(\mu)\zeta^n, \quad G(\zeta) = \sum_{n=0}^{\infty} d_n(\mu)\zeta^n, \quad \Psi(\zeta) = \sum_{n=0}^{\infty} \psi_n\zeta^n,
$$

where the coefficients ψ_n can be considered as known (see (3.19)), and we substituted the expansions in (3.29). This gives for $n = 0, 1, 2, \ldots$ the recursion relations

$$
\begin{aligned}
&(n+2)(n+1)c_{n+2} + (2n+1)d_n = \rho_n, \qquad && \rho_n = \sum_{k=0}^{n} \psi_k c_{n-k},\\
&(n+2)(n+1)d_{n+2} + 2\mu^4(n+1)c_{n+1} = \sigma_n, \qquad && \sigma_n = \sum_{k=0}^{n} \psi_k d_{n-k}.
\end{aligned}
\tag{3.31}
$$

If μ is large, the recursion relations cannot be solved in forward direction, because of numerical instabilities. For the Bessel function case we have shown that we can solve the system by iteration and backward recursion. The relation in (3.30) can be used for normalization of the coefficients in the backward recursion scheme.

For details we refer to [18]. The present case is identical to the case of the Bessel functions; only the function $\Psi(\zeta)$ is different, and instead of μ^2 in (3.31) we had the order ν of the Bessel functions.

4. Expansions from integral representations

The expansions developed by Olver, of which some are given in the previous sections, are all valid if $|a|$ is large. For several cases we gave modified expansions that hold if at least one of the two parameters a, z is large and we have indicated the relations between Olver's expansions and the new expansions. The modified expansions have in fact a double asymptotic property. Initially, we derived these expansions by using integral representations of the parabolic cylinder functions, and later we found the relations with Olver's expansions. In this section we explain how some of the modified expansions can be obtained from the integrals that define $U(a, z)$ and $V(a, z)$. Again we only consider real values of the parameters.

4.1 Expansions in terms of elementary functions by using integrals

4.1.1 The case $a \ge 0, z \ge 0;\quad a+z \gg 0.$ We start with the well-known integral representation

$$U(a,z) = \frac{e^{-\frac14 z^2}}{\Gamma(a+\frac12)} \int_0^\infty w^{a-\frac12} e^{-\frac12 w^2 - zw}\, dw, \quad a > -\tfrac12 \tag{4.1}$$

which we write in the form

$$U(a,z) = \frac{z^{a+\frac12} e^{-\frac14 z^2}}{\Gamma(a+\frac12)} \int_0^\infty w^{-\frac12} e^{-z^2\phi(w)}\, dw, \tag{4.2}$$

where

$$\phi(w) = w + \tfrac12 w^2 - \lambda \ln w, \quad \lambda = \frac{a}{z^2}. \tag{4.3}$$

The positive saddle point w_0 of the integrand in (4.3) is computed from

$$\frac{d\phi(w)}{dw} = \frac{w^2 + w - \lambda}{w} = 0, \tag{4.4}$$

giving

$$w_0 = \tfrac12 \left[\sqrt{1+4\lambda} - 1\right]. \tag{4.5}$$

We consider z as the large parameter. When λ is bounded away from 0 we can use Laplace's method (see [11] or [22]). When a and z are such that $\lambda \to 0$ Laplace's method cannot be applied. However, we can use a method given in [15] that allows small values of λ.

To obtain a standard form for this Laplace-type integral, we transform $w \to t$ (see [16]) by writing

$$\phi(w) = t - \lambda \ln t + A, \tag{4.6}$$

where A does not depend on t or w, and we prescribe that $w = 0$ should correspond with $t = 0$ and $w = w_0$ with $t = \lambda$, the saddle point in the $t-$plane.

This gives

$$U(a,z) = \frac{z^{a+\frac12} e^{-\frac14 z^2 - Az^2}}{(1+4\lambda)^{\frac14}\Gamma(a+\frac12)} \int_0^\infty t^{a-\frac12} e^{-z^2 t} f(t)\, dt, \tag{4.7}$$

where

$$f(t) = (1+4\lambda)^{\frac14} \sqrt{\frac{t}{w}}\, \frac{dw}{dt} = (1+4\lambda)^{\frac14} \sqrt{\frac{w}{t}}\, \frac{t-\lambda}{w^2 + w - \lambda}. \tag{4.8}$$

By normalizing with the quantity $(1+4\lambda)^{\frac14}$ we obtain $f(\lambda) = 1$, as can be verified from (4.8) and a limiting process (using l'Hôpital's rule). The quantity A is given by

$$A = \tfrac12 w_0^2 + w_0 - \lambda \ln w_0 - \lambda + \lambda \ln \lambda. \tag{4.9}$$

A first uniform expansion can be obtained by writing

$$f(t)=\sum_{n=0}^{\infty} a_n(\lambda)(t-\lambda)^n. \tag{4.10}$$

Details on the computation of $a_n(\lambda)$ will be given in the appendix.

By substituting (4.10) into (4.7) we obtain

$$U(a,z)\sim \frac{e^{-\frac14 z^2-Az^2}}{z^{a+\frac12}(1+4\lambda)^{\frac14}}\sum_{n=0}^{\infty} a_n(\lambda)P_n(a)z^{-2n}, \tag{4.11}$$

where

$$P_n(a)=\frac{z^{2a+2n+1}}{\Gamma(a+\frac12)}\int_0^\infty t^{a-\frac12}e^{-z^2t}(t-\lambda)^n\,dt,\quad n=0,1,2,\ldots\ . \tag{4.12}$$

The $P_n(a)$ are polynomials in a. They follow the recursion relation

$$P_{n+1}(a)=(n+\tfrac12)P_n(a)+anP_{n-1}(a),\quad n=0,1,2,\ldots\ ,$$

with initial values

$$P_0(a)=1,\quad P_1(a)=\tfrac12.$$

We can obtain a second expansion

$$U(a,z)\sim \frac{e^{-\frac14 z^2-Az^2}}{z^{a+\frac12}(1+4\lambda)^{\frac14}}\sum_{k=0}^{\infty}\frac{f_k(\lambda)}{z^{2k}}, \tag{4.13}$$

with the property that in the series the parameters λ and z are separated, by introducing a sequence of functions $\{f_k\}$ with $f_0(t)=f(t)$ and by defining

$$f_{k+1}(t)=\sqrt{t}\,\frac{d}{dt}\left[\sqrt{t}\,\frac{f_k(t)-f_k(\lambda)}{t-\lambda}\right],\quad k=0,1,2,\ldots\ . \tag{4.14}$$

The coefficients $f_k(\lambda)$ can be expressed in terms of the coefficients $a_n(\lambda)$ defined in (4.10). To verify this, we write

$$f_k(t)=\sum_{n=0}^{\infty} a_n^{(k)}(\lambda)(t-\lambda)^n, \tag{4.15}$$

and by substituting this in (4.14) it follows that

$$a_n^{(k+1)}(\lambda)=\lambda(n+1)a_{n+2}^{(k)}(\lambda)+\left(n+\tfrac12\right)a_{n+1}^{(k)}(\lambda),\quad k\ge0,\quad n\ge0. \tag{4.16}$$

Hence, the coefficients $f_k(\lambda)$ of (4.13) are given by

$$f_k(\lambda)=a_0^{(k)}(\lambda),\quad k\ge0. \tag{4.17}$$

We have

$$
\begin{array}{ll}
f_0(\lambda) & = 1, \\
f_1(\lambda) & = \frac{1}{2}\left[a_1(\lambda) + 2\lambda a_2(\lambda)\right], \\
f_2(\lambda) & = \frac{1}{4}\left[12\lambda^2 a_4(\lambda) + 14\lambda a_3(\lambda) + 3a_2(\lambda)\right], \\
f_3(\lambda) & = \frac{1}{8}\left[120\lambda^3 a_6(\lambda) + 220\lambda^2 a_5(\lambda) + 116\lambda a_4(\lambda) + 15a_3(\lambda)\right].
\end{array}
\tag{4.18}
$$

Explicitly,

$$
\begin{array}{ll}
f_0(\lambda) & = 1, \\
f_1(\lambda) & = -\frac{\rho}{24}(20\sigma^2 - 10\sigma - 1), \\
f_2(\lambda) & = \frac{\rho^2}{1152}(6160\sigma^4 - 6160\sigma^3 + 924\sigma^2 + 20\sigma + 1), \\
f_3(\lambda) & = -\frac{\rho^3}{414720}(27227200\sigma^6 - 40840800\sigma^5 + 16336320\sigma^4 - \\
& \quad 1315160\sigma^3 - 8112\sigma^2 + 2874\sigma + 1003),
\end{array}
\tag{4.19}
$$

where

$$
\sigma = \frac{1}{2}\left[1 + \frac{z}{\sqrt{4a+z^2}}\right], \quad \rho = \frac{(2\sigma-1)^2}{\sigma} = \frac{2z^2}{\sqrt{4a+z^2}(z+\sqrt{4a+z^2})}. \tag{4.20}
$$

We observe that $f_k(\lambda)$ is a polynomial of degree $2k$ in σ multiplied with ρ^k.

If a and z are positive then $\sigma \in [0,1]$. Furthermore, the sequence $\{\rho^k/z^{2k}\}$ is an asymptotic scale when one or both parameters a and z are large. The expansion in (4.13) is valid for $z \to \infty$ and holds uniformly for $a \geq 0$. It has a double asymptotic property in the sense that it is also valid as $a \to \infty$, uniformly with respect to $z \geq 0$. As follows from the coefficients given in (4.19) and relations to be given later, we can indeed let $z \to 0$ in the expansion.

The expansion in (4.13) can be obtained by using an integration by parts procedure. We give a few steps in this method. Consider the integral

$$
F_a(z) = \frac{1}{\Gamma(a+\frac{1}{2})}\int_0^\infty t^{a-\frac{1}{2}} e^{-z^2 t} f(t)\, dt, \tag{4.21}
$$

We have (with $\lambda = a/z^2$)

$$
\begin{aligned}
F_a(z) &= \frac{f(\lambda)}{\Gamma(a+\frac{1}{2})}\int_0^\infty t^{a-\frac{1}{2}} e^{-z^2 t}\, dt + \frac{1}{\Gamma(a+\frac{1}{2})}\int_0^\infty t^{a-\frac{1}{2}} e^{-z^2 t}[f(t)-f(\lambda)]\, dt \\
&= z^{-2a-1} f(\lambda) - \frac{1}{z^2\Gamma(a+\frac{1}{2})}\int_0^\infty t^{\frac{1}{2}} \frac{[f(t)-f(\lambda)]}{t-\lambda}\, de^{-z^2(t-\lambda\ln t)} \\
&= z^{-2a-1} f(\lambda) + \frac{1}{z^2\Gamma(a+\frac{1}{2})}\int_0^\infty t^{a-\frac{1}{2}} e^{-z^2 t} f_1(t)\, dt,
\end{aligned}
$$

where f_1 is given in (4.14) with $f_0 = f$. Repeating this procedure we obtain (4.13). More details on this method and proofs of the asymptotic nature of the expansions (4.11) and (4.13) can be found in our earlier papers. We concentrate on expansion (4.13) because (4.11) cannot be compared with Olver's expansions.

To compare (4.13) with Olver's expansion (2.16), we write

$$a = \tfrac{1}{2}\mu^2, \quad z = \mu\sqrt{2}t. \tag{4.22}$$

Then the parameters σ and ρ defined in (4.20) become

$$\sigma = \tfrac{1}{2}\left[1 + \frac{t}{\sqrt{1+t^2}}\right] = \widetilde{\tau} + 1, \quad \rho = \frac{2t^2}{\sqrt{1+t^2}(t+\sqrt{1+t^2})}, \tag{4.23}$$

where $\widetilde{\tau}$ is given in (2.32). After a few manipulations we write (4.13) in the form (cf.(2.29))

$$U\left(\tfrac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \frac{\widetilde{h}(\mu)\, e^{-\mu^2\widetilde{\xi}}}{(t^2+1)^{\frac{1}{4}}} \widetilde{F}_\mu(z), \quad \widetilde{F}_\mu(z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{\widetilde{\phi}_k(\sigma)}{\mu^{2k}}, \tag{4.24}$$

where

$$\widetilde{\xi} = \tfrac{1}{2}\left[t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)\right], \tag{4.25}$$

$$h(\mu) = e^{\frac{1}{4}\mu^2}\, \mu^{-\frac{1}{2}\mu^2 - \frac{1}{2}}\, 2^{\frac{1}{4}\mu^2 - \frac{1}{4}} \tag{4.26}$$

and

$$\widetilde{\phi}_k(\sigma) = \frac{(-1)^k}{(2t^2)^k} f_k(\lambda). \tag{4.27}$$

Explicitly,

$$\begin{aligned}
\widetilde{\phi}_0(\sigma) &= 1, \\
\widetilde{\phi}_1(\sigma) &= \frac{1-\sigma}{12}(20\sigma^2 - 10\sigma - 1), \\
\widetilde{\phi}_2(\sigma) &= \frac{(1-\sigma)^2}{288}(6160\sigma^4 - 6160\sigma^3 + 924\sigma^2 + 20\sigma + 1), \\
\widetilde{\phi}_3(\sigma) &= \frac{(1-\sigma)^3}{51840}(27227200\sigma^6 - 40840800\sigma^5 + 16336320\sigma^4 - \\
&\quad 1315160\sigma^3 - 8112\sigma^2 + 2874\sigma + 1003),
\end{aligned} \tag{4.28}$$

where σ is given in (4.23). Comparing (4.24) with (2.29) we obtain $\widetilde{\phi}_k(\sigma) = \phi_k(\widetilde{\tau}), k \geq 0$, because $\sigma = 1 + \widetilde{\tau}$.

4.1.2 The case $a \geq 0, z \leq 0; \quad a - z \gg 0$. To derive the first expansion in (2.34) we use the contour integral

$$U(a,-z) = \frac{\sqrt{2\pi}e^{\frac{1}{4}z^2}}{\Gamma(a+\frac{1}{2})} H_a(z), \quad H_a(z) = \frac{\Gamma(a+\frac{1}{2})}{2\pi i} \int_{\mathcal{C}} e^{zs + \frac{1}{2}s^2}\, s^{-a-\frac{1}{2}}\, ds, \tag{4.29}$$

where $\mathcal{C}$ is a vertical line in the half plane $\Re s > 0$. This integral can be transformed into a standard form that involves the same mapping as in the previous subsection. We first write (by transforming via $s = zw$)

$$H_a(z) = \frac{z^{\frac12 - a}\Gamma(a+\frac12)}{2\pi i}\int_{\mathcal{C}} e^{z^2(w+\frac12 w^2)} w^{-a-\frac12}\,dw = \frac{z^{\frac12 - a}\Gamma(a+\frac12)}{2\pi i}\int_{\mathcal{C}} e^{z^2\phi(w)}\,\frac{dw}{\sqrt{w}}, \quad (4.30)$$

where $\phi(w)$ is defined in (4.3). By using the transformation given in (4.6) it follows that

$$H_a(z) = \frac{z^{\frac12 - a}\Gamma(a+\frac12)e^{Az^2}}{2\pi i}\int_{\mathcal{C}} e^{z^2 t} t^{-a-\frac12} f(t)\,dt. \quad (4.31)$$

The integration by parts method used for (4.21) gives the expansion (see [15])

$$H_a(z) \sim \frac{z^a e^{Az^2}}{(4a+z^2)^{\frac14}}\sum_{k=0}^{\infty}(-1)^k\frac{f_k(\lambda)}{z^{2k}}, \quad (4.32)$$

where the $f_k(\lambda)$ are the same as in (4.13); see also (4.18). This gives the first expansion of (2.34).

Remark 4.1. The first result in (2.34) can also be obtained by using (4.1) with $z < 0$. The integral for $U(a, -z)$ can be written as in (4.2), now with $\phi(w) = \frac12 w^2 - w - \ln\lambda,\ \lambda = a/z^2$. In this case the relevant saddle point at $w_0 = (1+\sqrt{1+4\lambda})/2$ is always inside the interval $[1,\infty)$ and the standard method of Laplace can be used. The same expansion will be obtained with the same structure and coefficients as in (2.34), because of the unicity of Poincaré-type asymptotic expansions. See also Section 4.1.4 where Laplace's method will be used for an integral that defines $V(a, z)$.

4.1.3 The case $a \le 0, z > 2\sqrt{-a},\quad -a + z \gg 0$. Olver's starting point (2.1) can also be obtained from an integral. Observe that (4.1) is not valid for $a \le -\frac12$. We take as integral (see [1], p. 687, 19.5.1)

$$U(-a,z) = \frac{\Gamma(\frac12 + a)}{2\pi i} e^{-\frac14 z^2}\int_{\alpha} e^{zs - \frac12 s^2} s^{-a-\frac12}\,ds, \quad (4.33)$$

where α is a contour that encircles the negative $s-$axis in positive direction. Using a transformation we can write this in the form (cf. (4.2))

$$U(-a,z) = \frac{\Gamma(\frac12 + a)}{2\pi i} z^{\frac12 - a} e^{-\frac14 z^2}\int_{\alpha} e^{\phi(w)} w^{-\frac12}\,dw, \quad (4.34)$$

where

$$\phi(w) = w - \tfrac12 w^2 - \lambda\ln w, \quad \lambda = \frac{a}{z^2}. \quad (4.35)$$

The relevant saddle point is now given by

$$w_0 = \tfrac12\left(1 - \sqrt{1-4\lambda}\right), \quad 0 < \lambda < \tfrac14. \quad (4.36)$$

When $\lambda \to 0$ the standard saddle point method is not applicable, and we can again use the methods of our earlier papers ([15] and [16]) and transform

$$\phi(w) = t - \lambda \ln t + A, \tag{4.37}$$

where the points at $-\infty$ in the $w-$ and $t-$ plane should correspond, and $w = w_0$ with $t = \lambda$. We obtain

$$U(-a,z) = \frac{\Gamma(\frac{1}{2}+a)}{(1-4\lambda)^{\frac{1}{4}}\, 2\pi i} z^{\frac{1}{2}-a} e^{-\frac{1}{4}z^2 + z^2 A} \int_\alpha e^{z^2 t}\, t^{-a-\frac{1}{2}} f(t)\, dt, \tag{4.38}$$

where α is a contour that encircles the negative $t-$axis in positive direction and

$$f(t) = (1-4\lambda)^{\frac{1}{4}} \sqrt{\frac{t}{w}} \frac{dw}{dt} = (1-4\lambda)^{\frac{1}{4}} \sqrt{\frac{w}{t}} \frac{t-\lambda}{w - w^2 - \lambda}. \tag{4.39}$$

Expanding $f(t)$ as in (4.10), and computing $f_k(\lambda)$ as in the procedure that yields the relations in (4.18), we find that the same values $f_k(\lambda)$ as in (4.19), up to a factor$(-1)^k$ and a different value of τ and ρ. By using the integration by parts method for contour integrals [15], that earlier produced (4.32), we obtain the result

$$U(-a,z) \sim \frac{z^a\, e^{Az^2 - \frac{1}{4}z^2}}{(z^2-4a)^{\frac{1}{4}}} \sum_{k=0}^{\infty} (-1)^k \frac{f_k(\lambda)}{z^{2k}}, \tag{4.40}$$

where the first $f_k(\lambda)$ are given in (4.19) with

$$\sigma = \tfrac{1}{2}\left[1 + \frac{z}{\sqrt{z^2-4a}}\right], \quad \rho = \frac{(2\sigma-1)^2}{\sigma} = \frac{2z^2}{\sqrt{z^2-4a}+(z+\sqrt{z^2-4a})}. \tag{4.41}$$

This expansion can be written in the form (2.9).

4.1.4 The case $\quad a \le 0, z < -2\sqrt{-a}, \quad -a - z \gg 0.$ We use the relation (see (1.7))

$$U(-a,-z) = \sin \pi a\, U(-a,z) + \frac{\pi}{\Gamma(\frac{1}{2}-a)} V(-a,z), \tag{4.42}$$

and use the result of $U(-a,z)$ given in (4.40) or the form (2.9). An expansion for $V(-a,z)$ in (4.42) can be obtained from the integral (see [9])

$$V(a,z) = \frac{e^{-\frac{1}{4}z^2}}{2\pi} \int_{\gamma_1 \cup \gamma_2} e^{-\frac{1}{2}s^2 + zs}\, s^{a-\frac{1}{2}}\, ds, \tag{4.43}$$

where γ_1 and γ_2 are two horizontal lines, γ_1 in the upper half plane $\Im s > 0$ and γ_2 in the lower half plane $\Im s < 0$; the integration is from $\Re s = -\infty$ to $\Re s = +\infty$. (Observe that when we integrate on γ_1 in the other direction (from $\Re s = +\infty$ to $\Re s = \infty$) the contour $\gamma_1 \cup \gamma_2$ can be deformed into α of (4.33), and the integral defines $U(a,z)$, up to a factor.) We can apply Laplace's method to obtain the expansion given in (2.14) (see Remark 4.1).

4.2 The singular points of the mapping (4.6)
The mapping defined in (4.6) is singular at the saddle point

$$w_- = -\tfrac{1}{2}\left(\sqrt{1+4\lambda}+1\right). \tag{4.44}$$

If $\lambda = 0$ then $w_- = -1$ and the corresponding $t-$value is $-\frac{1}{2}$. For large values of λ we have the estimate:

$$t(w_-) \sim \lambda\left[-0.2785 - \frac{0.4356}{\sqrt{\lambda}}\right]. \tag{4.45}$$

This estimate is obtained as follows. The value $t_- = t(w_-)$ is implicitly defined by equation (4.6) with $w = w_-$. This gives

$$\begin{aligned} t_- - \lambda \ln t_- - \lambda + \lambda \ln \lambda &= -\tfrac{1}{2}\sqrt{1+4\lambda} \pm \lambda\pi i + \lambda \ln \frac{4\lambda}{(1+\sqrt{1+4\lambda})^2} \\ &= \pm\lambda\pi i - 2\sqrt{\lambda}\left[1 + \frac{1}{24\lambda} + \mathcal{O}(\lambda^{-2})\right], \end{aligned} \tag{4.46}$$

as $\lambda \to \infty$. The numerical solution of the equation $s - \ln s - 1 = \pm\pi i$ is given by $s_\pm = 0.2785\cdots e^{\mp\pi i}$. This gives the leading term in (4.46). The other term follows by a further simple step.

4.3 Expansions in terms of Airy functions
All results for the modified Airy-type expansions given in Section 3.3 can be obtained by using certain loop integrals. The integrals in (4.33) and (4.43) can be used for obtaining (3.23) and (3.24), respectively. The method is based on replacing $\phi(w)$ in (4.34) by a cubic polynomial, in order to take into account the influence of both saddle points of $\phi(w)$. This method is first described in [6]; see also [11] and [22].

5. Numerical verifications

We verify several asymptotic expansions by computing the error in the Wronskian relation for the series in the asymptotic expansions. Consider Olver's expansions of Section 2.3 for the oscillatory region $-1 < t < 1$ with negative a. We verify the relation in (2.28). Denote the left-hand side of the first line in (2.28) by $W(\mu, t)$. Then we define as the error in the expansions

$$\Delta(\mu, t) := \left| \frac{W(\mu,t)}{1 - \dfrac{1}{576\mu^4} + \dfrac{2021}{2488320\mu^8}} - 1 \right| \tag{5.1}$$

Taking 3 terms in the series of (2.23), (2.24) and (2.27), we obtain for several values of μ and t the results given in Table 5.1. We clearly see the loss of accuracy when t is close to 1. Exactly the same results are obtained for negative values of t in this interval.

t \ μ	5	10	25	50	100
.00	.32e-09	.78e-13	.13e-17	.32e-21	.78e-25
.10	.26e-09	.63e-13	.11e-17	.26e-21	.63e-25
.20	.81e-10	.20e-13	.33e-18	.82e-22	.20e-25
.30	.16e-08	.39e-12	.65e-17	.16e-20	.39e-24
.40	.88e-08	.22e-11	.36e-16	.89e-20	.22e-23
.50	.51e-07	.13e-10	.21e-15	.52e-19	.13e-22
.60	.40e-06	.99e-10	.17e-14	.40e-18	.99e-22
.70	.53e-05	.13e-08	.22e-13	.54e-17	.13e-20
.80	.20e-03	.50e-07	.84e-12	.20e-15	.50e-19
.90	.35e-00	.24e-04	.41e-09	.10e-12	.25e-16

Table 5.1. Relative accuracy $\Delta(\mu,t)$ defined in (5.1) for the asymptotic series of Section 2.3.

Next we consider the modified expansions of Section 2.1. Denote the left-hand side of (2.20) by $W(\mu,t)$. Then we define as the error in the expansions

$$\Delta(\mu,t) := \left|\tfrac{1}{2}W(\mu,t) - 1\right| \tag{5.2}$$

When we use the series in (2.9), (2.14), (2.18) and (2.19) with 5 terms, we obtain the results given in Table 5.2. We observe that the accuracy improves as μ or t increase. This shows the double asymptotic poperty of the modified expansions of Section 2.1.

t \ μ	5	10	25	50	100
1.1	.51e-01	.48e-05	.72e-10	.18e-13	.43e-17
1.2	.39e-04	.79e-08	.13e-12	.32e-16	.78e-20
1.3	.83e-06	.19e-09	.32e-14	.78e-18	.19e-21
1.4	.56e-07	.13e-10	.23e-15	.55e-19	.13e-22
1.5	.71e-08	.17e-11	.29e-16	.70e-20	.17e-23
2.0	.10e-10	.25e-14	.43e-19	.10e-22	.25e-26
2.5	.21e-12	.52e-16	.87e-21	.21e-24	.52e-28
5.0	.12e-16	.28e-20	.48e-25	.12e-28	.28e-32
10.0	.20e-20	.48e-24	.81e-29	.20e-32	.48e-36
25.0	.30e-25	.73e-29	.12e-33	.30e-37	.73e-41

Table 5.2. Relative accuracy $\Delta(\mu,t)$ defined in (5.2) for the asymptotic series of Section 2.1.

Finally we consider the expansions of Sections 2.4 and 2.5. Let the left-hand side of (2.35) be denoted by $W(\mu,t)$. Then we define as the error in the expansions

$$\Delta(\mu,t) := \left|\tfrac{1}{2}W(\mu,t) - 1\right| \tag{5.3}$$

When we use the series in (2.29), (2.33) and (2.34) with 5 terms, we obtain the results of Table 5.3. We again observe that the accuracy improves as μ or t increase. This shows the double asymptotic poperty of the modified expansions of Sections 2.4 and 2.5.

t \ μ	5	10	25	50	100
.00	.32e-09	.78e-13	.13e-17	.32e-21	.78e-25
.25	.12e-09	.28e-13	.47e-18	.12e-21	.28e-25
.50	.45e-11	.11e-14	.19e-19	.46e-23	.11e-26
.75	.57e-11	.14e-14	.24e-19	.58e-23	.14e-26
1.0	.27e-11	.65e-15	.11e-19	.27e-23	.65e-27
1.5	.29e-13	.70e-17	.12e-21	.29e-25	.70e-29
2.0	.20e-13	.48e-17	.81e-22	.20e-25	.48e-29
2.5	.43e-14	.11e-17	.18e-22	.43e-26	.11e-29
5.0	.45e-17	.11e-20	.18e-25	.45e-29	.11e-32
10.0	.16e-20	.38e-24	.64e-29	.16e-32	.38e-36

Table 5.3. Relative accuracy $\Delta(\mu, t)$ defined in (5.3) for the asymptotic series of Sections 2.4 and 2.5.

6. Concluding remarks

As mentioned in Section 1.1 of the Introduction, several sources for numerical algorithms for evaluating parabolic cylinder functions are available in the literature, but not so many algorithms make use of asymptotic expansions. The paper [10] is a rich source for asymptotic expansions, for all combinations of real and complex parameters, where always $|a|$ has to be large. There are no published algorithms that make use of Olver's expansions, although very efficient algorithms can be designed by using the variety of these expansions; [3] is the only reference we found in which Olver's expansions are used for numerical computations.

We started our efforts in making algorithms for the case of real parameters. We selected appropriate expansions from Olver's paper and for some cases we modified Olver's expansions in order to get expansions having a double asymptotic property. A serious point is making efficient use of the powerful Airy-type expansions that are valid near the turning points of the differential equation (and in much larger intervals and domains of the complex plane). In particular, constructing reliable software for all possible combinations of the complex parameters a and z is a challengeing problem.

A point of research interest is also the construction of error bounds for Olver's expansions and the modified expansions. Olver's paper is written before he developed the construction of bounds for the remainders, which he based on methods for differential equations, and which are available now in his book [11].

7. Appendix: Computing the coefficients $f_k(\lambda)$ of (4.13)

We give the details on the computation of the coefficients $f_k(\lambda)$ that are used in (4.13). The first step is to obtain coefficients d_k in the expansion

$$w = d_0 + d_1(t-\lambda) + d_2(t-\lambda)^2 + \dots, \tag{7.1}$$

where $d_0 = w_0$. From (4.6) we obtain

$$\frac{dw}{dt} = \frac{w}{t}\,\frac{t-\lambda}{w^2+w-\lambda}. \tag{7.2}$$

Substituting (7.1) we obtain

$$d_1^2 = \frac{w_0}{\lambda(1+2w_0)}, \tag{7.3}$$

where the saddle point w_0 is defined in (4.5). From the conditions on the mapping (4.6) it follows that $d_1 > 0$. Higher order coefficients d_k can be obtained from the first ones by recursion.

When we have determined the coefficients in (7.1) we can use (4.8) to obtain the coefficients $a_n(\lambda)$ of (4.10).

For computing in this way a set of coefficients $f_k(\lambda)$, say $f_0(\lambda), \ldots, f_{15}(\lambda)$, we need more than 35 coefficients d_k in (7.1). Just taking the square root in (7.3) gives for higher coefficients d_k very complicated expressions, and even by using computer algebra programs, as Maple, we need suitable methods in computing the coefficients.

The computation of the coefficients $d_k, a_n(\lambda)$ and $f_k(\lambda)$ is done with a new parameter $\theta \in [0, \frac{1}{2}\pi)$ which is defined by

$$4\lambda = \tan^2\theta. \tag{7.4}$$

We also write

$$\sigma = \cos^2 \tfrac{1}{2}\theta, \tag{7.5}$$

which is introduced earlier in (4.20) and (4.23). Then

$$w_0 = \frac{1-\sigma}{2\sigma-1}, \quad \lambda = \frac{\sigma(1-\sigma)}{(2\sigma-1)^2}, \quad d_1 = \frac{2\sigma-1}{\sigma}. \tag{7.6}$$

In particular the expressions for w_0 and d_1 are quite convenient, because we can proceed without square roots in the computations. Higher coefficients d_k can be obtained by using (7.2).

The first relation $f_0(\lambda) = a_0^{(0)}(\lambda) = 1$ easily follows from (4.3), (4.8), (7.1) and (7.6):

$$f_0(\lambda) = (1+4\lambda)^{\frac{1}{4}}\sqrt{\frac{\lambda}{w_0}}\, d_1 = 1.$$

Then using (4.8) we obtain

$$a_0(\lambda) = 1,\ a_1(\lambda) = -\frac{\cos^2\theta(1+2c)^2}{6(c+1)c^2},\ a_2(\lambda) = \frac{\cos^4\theta(20c^4+40c^3+30c^2+12c+3)}{24(c+1)^2c^4},$$

where $c = \sqrt{\sigma} = \cos\frac{1}{2}\theta$. Using the scheme leading to (4.17) one obtains the coefficients $f_k(\lambda)$. The first few coefficients are given in (4.19).

We observe that $f_k(\lambda)$ is a polynomial of degree $2k$ in σ multiplied with ρ^k. If a and z are positive then $\sigma \in [0,1]$. It follows that the sequence $\{\rho^k/z^{2k}\}$ is an asymptotic scale when one or both parameters a and z are large, and, hence, that $\{f_k(\lambda)/z^{2k}\}$ of (4.13) is an asymptotic scale when one or both parameters a and z are large.

Because of the relation in (4.27) and $\widetilde{\phi}_k(\sigma) = \phi_k(\widetilde{\tau})$, higher coefficients $f_k(\lambda)$ can also be obtained from the recursion relation (2.11), which is obtained by using the differential equation of the parabolic cylinder functions.

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